IUCAA Refresher Course 2026 SK May 19–25, 2026

Lectures on Basics of Gravitational Waves. (1)

1. Introductory slides :

(a) GW150914 plot of Strain vs 't'

Note value on y-axis 10^{-21} (dimensionless number).

y-axis → 'strain' $\frac{\Delta L}{L}$

x-axis → 'time'

Exceptionally 'small' number '10^{-21}'

(b) Strain related to $h(t)$ → 'metric perturbation'

Note one is 'seeing' the metric directly.

(c) Plot is of a 'signal' which is seen by a 'detector' (LIGO).

Signal → from where ? origins ? Theory ?

Detector → sensitivity

↓

Data → analysis of data → extraction of signal.

Three aspects → (1) GR or theory of gravity

(2) Interferometric detection

(3) Data analysis.

2. Earlier detection : Cumulative periastron shift in binary pulsars PSRB-1913+16 and later other similar systems.

Highly accurate measurements.

3. Brief summary of history

– timeline of important discoveries.

Recap of selected topics in GR.

1. Geodesics.

$x^i \equiv x^i(\tau)$

Geodesic equ
$$\frac{d^2 x^i}{d\tau^2} + \Gamma^i_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} = 0 \qquad (1)$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{im}\left[g_{mj,k} + g_{mk,j} - g_{jk,m}\right] \qquad \text{'Christofel symbol'}$$

$$g_{ij}\dot{x}^i\dot{x}^j = K, \quad \left.\begin{array}{ll} K=-1 & \text{timelike} \\ K=0 & \text{null}\end{array}\right\} \qquad \dot{x}^i = \frac{dx^i}{d\tau}$$

Affine parametrisation for timelike trajectories.

Equ (1) follows from $S = \int_A^B \sqrt{g_{ij}\dot{x}^i\dot{x}^j}\, d\tau$, 'variation w.r.t x^i'

where $s \sim \tau$ (arclength, affine parametrisation).

$g_{ij}\dot{x}^i\dot{x}^j = K \longrightarrow$ first integral of (1) for $K=0,-1$.

Action for null geodesics $S' = \int_A^B (g_{ij}\dot{x}^i\dot{x}^j)\, d\tau$

Variation w.r.t x^i gives (1) directly.

Timelike / null condition has to be imposed.

We will be concerned with metrics of the form

$$g_{ij} = \eta_{ij} + h_{ij} \quad \text{where } \|h_{ij}\| \text{ is small.}$$

Γ^i_{jk} to linear order in h_{ij} will be

$$\Gamma^i_{jk} = \frac{1}{2}\eta^{im}\left[h_{mj,k} + h_{mk,j} - h_{jk,m}\right]$$

Note that $g^{ij} = \eta^{ij} - h^{ij}$ in order to maintain $g_{ij}g^{jk} \cong \delta^i_k$ to linear order in 'h'.

Newtonian gravity can be rewritten, as follows:

Take $t = a\tau + b$ $\quad \frac{dt}{d\tau} = a \quad \frac{d^2t}{d\tau^2} = 0$ (2)

$$\frac{d^2x^\alpha}{dt^2} = -\frac{\partial\phi}{\partial x^\beta}\delta^{\alpha\beta} \qquad \text{Newton's 2nd law}$$

$$\frac{1}{a^2}\frac{d^2x^\alpha}{d\tau^2} = -\frac{\partial\phi}{\partial x^\beta}\delta^{\alpha\beta} \Rightarrow \frac{d^2x^\alpha}{d\tau^2} + \frac{\partial\phi}{\partial x^\beta}\delta^{\alpha\beta}a^2 = 0$$

$$\Rightarrow \frac{d^2x^\alpha}{d\tau^2} + \frac{\partial\phi}{\partial x^\beta}\delta^{\alpha\beta}\left(\frac{dt}{d\tau}\right)^2 = 0 \qquad (3)$$

Compare (2) and (3) with

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} = 0$$

Taking $x^0 = t$ $\quad \frac{d^2t}{d\tau^2} = 0 \Rightarrow \Gamma^0_{jk} = 0$

x^α $(\alpha = 1,2,3)$ $\quad \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} = 0.$

$$\Rightarrow \Gamma^\alpha_{00} = \frac{\partial\phi}{\partial x^\beta}\delta^{\alpha\beta}.$$

Thus, $\frac{d^2x^\alpha}{dt^2} = -\Gamma^\alpha_{00}$ is Newton's 2nd law

This is the Cartan formulation of Newtonian gravity.

2. Geodesic deviation.

Newtonian deviation eqn : $\frac{d^2\eta^\alpha}{dt} = -\delta^{\alpha\beta}\frac{\partial^2\phi}{\partial x^\beta\partial x^\gamma}\eta^\gamma.$

$\eta^\alpha \to$ deviation vector

Derivation : write Newton's 2nd law for x^α, $x^\alpha + \eta^\alpha$ and subtract the eqn for x^α

Taylor expand $\frac{\partial\phi}{\partial x^\alpha}$ from the eqn for $x^\alpha + \eta^\alpha$.

$$-\frac{\partial^2\phi}{\partial x^\beta\partial x^\gamma} = \mathcal{E}_{\beta\gamma} \to \text{'Tidal tensor' (Newtonian).}$$

Geodesic deviation in Riemannian and pseudo-Riemannian

Space / spacetimes :

$$(u^j \nabla_j)(u^k \nabla_k)\eta^m = -R^m{}_{pqr} u^p u^r \eta^q$$

where η^m is the deviation 4-vector.

Note the appearance of the Riemann tensor here.

Geodesic deviation eqns correspond to a set of linear ODES Coupled and with variable coefficients. Usually very tough to solve. Note geodesic deviation is 'perturbative' (small η^i).

Why are we studying geodesics / geodesic deviation in this Course? What is the importance.

Geodesics represent freely falling particle trajectories. The mirrors, beam splitters in interferometers are to be taken as on 'geodesics'. The separation is the 'deviation'. Change in deviation is caused by the impinging GW. We need to keep this picture in mind. But we need to remember this is 'approximate' but 'very close' to reality. Surely it is not 'exact'.

For $g_{ij} = \eta_{ij} + h_{ij}$, $u^i = (1,0,0,0)$, the deviation eqn becomes simple.

$$(u^i \partial_i)(u^j \partial_j)\eta^k = -R^k{}_{0q0}\eta^q$$

$$\Rightarrow \quad \frac{d^2\eta^k}{d\tau^2} = -R^k{}_{0q0}\eta^q .$$

We will use this later.

Killing vectors, isometries.

$$x^i \to x^i + \xi^i \qquad \|\xi^i\| \ll 1 \quad \text{'small'}$$

Infinitesimal coordinate transformation.

Recall $\quad g_{ij}(x) = \dfrac{\partial x'^m}{\partial x^i} \dfrac{\partial x'^n}{\partial x^j} g'_{mn}(x')$

$$\Rightarrow \quad g_{ij}(x) = \left(\delta^m_i + \partial_i \xi^m\right)\left(\delta^n_j + \partial_j \xi^n\right) g'_{mn}(x').$$

$$= g'_{ij}(x') + (\partial_i \xi^m)\, g'_{mj}(x') + (\partial_j \xi^n)\, g'_{in}(x') + O(\xi^2).$$

$$= g'_{ij}(x) + \xi^k \partial_k g'_{ij}(x) + (\partial_i \xi^m)\, g'_{mj}(x') + (\partial_j \xi^n)\, g'_{in}(x') + O(\xi^2) \qquad (4)$$

Form invariance $\quad g'_{ij}(x) = g_{ij}(x) \Rightarrow$

$$\xi^k \partial_k g_{ij}(x) + (\partial_i \xi^m)\, g_{mj}(x) + (\partial_j \xi^n)\, g_{in}(x) = 0$$

This becomes. $\quad \nabla_i \xi_j + \nabla_j \xi_i = 0 \to$ Killing eqn.

The vector field which satisfies this eqn, $\xi^i(x)$ constitutes the Killing vector field.

Use $g_{ij} = (\eta_{ij} + h_{ij})$ $\quad g'_{ij} = (\eta_{ij} + h'_{ij})$ in (4)

$$h_{ij} = h'_{ij} + (\partial_i \xi^m)\, \eta_{mj} + (\partial_j \xi^n)\, \eta_{in}$$

(to linear order in small quantities)

$$\Rightarrow \quad h'_{ij} = h_{ij} - \partial_i \xi_j - \partial_j \xi_i$$

We will use this eqn later while analysing gauge freedom and gauge choices.

The Kerr metric.

Vacuum, axisymmetric, stationary soln of $R_{ij} = 0$

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 - \frac{4Mra \sin^2\theta}{\rho^2} d\phi\, dt$$

$$+ \frac{\Sigma \sin^2\theta}{\rho^2} d\phi\, dt$$

where

$$\rho^2 = r^2 + a^2 \cos^2\theta$$

$$\Delta = r^2 + a^2 - 2Mr$$

$$\Sigma = (r^2+a^2)^2 - a^2 \Delta \sin^2\theta .$$

M, a are two parameters. $a \to$ rotation. parameter.

$a = 0 \longrightarrow$ Schwarzschild metric,

$M = 0 \longrightarrow$ flat spacetime in oblate spheroidal coordinates.

Slow rotation : 'a^2' terms ignored

The term $- \dfrac{4Mra \sin^2\theta}{\rho^2}$ remains and signals the rotation.

3 different observers : (a) ZAMO (Zero Angular Mom. Obs.).

Constant of motion $L = 0$, still an nonzero $\dfrac{d\phi}{dt} = \omega$

$\longrightarrow$ frame dragging

(b) $t^i = (1, 0, 0, 0)$ static obs. Nullness of $g_{ij} t^i t^j$

sets the boundary $r = r_{se} = M + \sqrt{M^2 - a^2 \cos^2\theta}$.

(c) $\phi^i = (1, 0, 0, \Omega)$ stationary obs. Nullness of $g_{ij} \phi^i \phi^j$

sets the boundary $r = r_H = M + \sqrt{M^2 - a^2}$ (event horizon).

$M > a \rightarrow$ requirement of existence of horizon.

$M = a \rightarrow$ extremal Kerr.

If $M < a \rightarrow$ naked singularity (no horizon).

Kerr metric represents a rotating eternal black hole.

Note: For Schwarzschild there is a interior soln.
For Kerr the interior problem is still not fully solved.

Linearised gravity:

$\rightarrow$ energy momentum tensor.

Einstein's eqns $\quad G_{ij} = \frac{8\pi G}{c^4} T_{ij}$

$\downarrow$

Einstein tensor $\quad R_{ij} - \frac{1}{2} g_{ij} R$

We assume $g_{ij} = \eta_{ij} + h_{ij}$, keep terms upto linear order in 'h'. This will give us the linearised Einstein equations. Let us find the eqns.

Start with the Riemann tensor.

$$R^i{}_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{km} \Gamma^m_{jl} - \Gamma^i_{lm} \Gamma^m_{jk}.$$

The $\Gamma\Gamma$ terms will contain $h^2 \rightarrow$ so we need not bother about them in linearised gravity (while constructing the field eqns).

Ricci tensor:

$$R^i{}_{jil} = \partial_i \Gamma^i_{jl} - \partial_l \Gamma^i_{ji} + \Gamma^i_{im} \Gamma^m_{jl} - \Gamma^i_{lm} \Gamma^m_{ji}$$

$$R^{i}{}_{jkl} \simeq \partial_k \Gamma^{i}_{jl} - \partial_l \Gamma^{i}_{jk}$$

Use $\Gamma^{i}_{jl} = \frac{1}{2} \eta^{im} \left(h_{mj,l} + h_{ml,j} - h_{jl,m} \right)$

$$\therefore \quad R^{i}{}_{jkl} = \frac{1}{2} \eta^{im} \Big[\cancel{\partial_k \partial_l h_{mj}} + \partial_k \partial_j h_{ml} - \partial_k \partial_m h_{jl}$$

$$- \cancel{\partial_l \partial_k h_{mj}} - \partial_l \partial_j h_{mk} + \partial_l \partial_m h_{jk} \Big]$$

$$= \frac{1}{2} \eta^{im} \left[\partial_k \partial_j h_{ml} - \partial_k \partial_m h_{jl} - \partial_l \partial_j h_{mk} + \partial_l \partial_m h_{jk} \right]$$

$$R_{jl} = \frac{1}{2} \eta^{im} \left[\partial_i \partial_j h_{ml} - \partial_i \partial_m h_{jl} - \partial_l \partial_j h_{mi} + \partial_l \partial_m h_{ji} \right]$$

$$= \frac{1}{2} \left[\partial^m \partial_j h_{ml} - \Box h_{jl} - \partial_l \partial_j h + \partial_l \partial^i h_{ji} \right]$$

$$R = \frac{1}{2} \left[\partial^m \partial^l h_{ml} - \Box h - \Box h + \partial^j \partial^i h_{ji} \right]$$

$$= \left(\partial^m \partial^l h_{ml} - \Box h \right)$$

$$G_{jl} = R_{jl} - \frac{1}{2} \eta_{jl} R .$$

$$= \frac{1}{2} \left[\partial^m \partial_j h_{ml} - \Box h_{jl} - \partial_l \partial_j h + \partial_l \partial^i h_{ji} \right]$$

$$- \frac{1}{2} \eta_{jl} \left[\partial^m \partial^n h_{mn} - \Box h \right]$$

Introduce $\bar{h}_{jl} = h_{jl} - \frac{1}{2} \eta_{jl} h$. (Trace reversed metric perturbation)

$$\mathrm{Tr}\, \bar{h}_{jl} = - h$$

This gives. $h_{\nu\rho} = \bar{h}_{\nu\rho} + \frac{1}{2}\eta_{\nu\rho} h.$

$$G_{\nu\rho} = -\frac{1}{2}\Box \bar{h}_{\nu\rho} - \frac{1}{4}\eta_{\nu\rho}\Box h + \frac{1}{2}\partial^m \cancel{\partial_\nu \left(\bar{h}_{m\rho} + \frac{1}{2}\eta_{m\rho} h\right)}$$

$$-\frac{1}{2}\partial_\rho \partial_\nu h + \frac{1}{2}\partial_\rho \cancel{\partial^i \left(\bar{h}_{\nu i} + \frac{1}{2}\eta_{\nu i} h\right)}$$

$$-\frac{1}{2}\eta_{\nu\rho}\left[\partial^m \cancel{\partial^n \left(\bar{h}_{mn} + \frac{1}{2}\eta_{mn} h\right)}\right]$$

$$+\frac{1}{2}\eta_{\nu\rho}\Box h.$$

Use the gauge condition (de Donder, Lorentz) $\partial^i \bar{h}_{\nu i} = 0$

$$G_{\nu\rho} = -\frac{1}{2}\Box \bar{h}_{\nu\rho} - \cancel{\frac{1}{4}\eta_{\nu\rho}\Box h} + \cancel{\frac{1}{4}\partial_\rho\partial_\nu h} - \cancel{\frac{1}{2}\partial_\rho\partial_\nu h}$$

$$+\cancel{\frac{1}{4}\partial_\rho\partial_\nu h} - \cancel{\frac{1}{4}\eta_{\nu\rho}\Box h} + \cancel{\frac{1}{2}\eta_{\nu\rho}\Box h}.$$

$$= -\frac{1}{2}\Box \bar{h}_{\nu\rho}.$$

Thus, we get the Einstein eqns as

$$-\frac{1}{2}\Box \bar{h}_{\nu\rho} = \frac{8\pi G}{c^4} T_{\nu\rho}.$$

This is the linearised Einstein equation.

The trace gives $-\frac{1}{2}\Box h = \frac{8\pi G}{c^4} T.$

In vacuum $\Box \bar{h}_{\nu\rho} = 0$

$$\Rightarrow \Box h_{\nu\rho} = 0 \quad \text{since } \Box h = 0.$$

Much of what we will do rests on this linearised eqn.

In a curved background

$g_{ij} = g^{B}_{ij} + h_{ij}$ the eqn takes a different form using the Lorentz gauge $\nabla^{i} \bar{h}_{ij} = 0$. In vacuum we end up with.

$$\Box h_{ik} + 2 R_{ijkl} h^{jl} = 0$$

where the $\Box \equiv g^{ij} \nabla_i \nabla_j$ 'Covariant derivative'

Note the appearance of the Riemann tensor. It appears because of the non-commutation of the covariant derivative.

For derivation see Flanagan-Hughes or lectures by Scott Hughes or other books/chapters. mentioned in the list of references.

Q What are the solns of $\Box \bar{h}_{ij} = -\frac{16\pi G}{c^4} T_{ij}$

in vacuum and in matter.

$T_{ij} = 0 \Rightarrow$ essentially tells us about the propagating wave in vacuum

$T_{ij} \neq 0 \Rightarrow$ addresses the generation problem and how waves are created given a source.